# Perfectly Matched Layers and Harmoniously Matched Layers: a numerical comparison for 2D acoustic propagation in heterogeneous media

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**Perfectly Matched Layers and Harmoniously Matched Layers: a numerical comparison for 2D acoustic propagation in heterogeneous media**

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**Abstract**

The Harmoniously Matched Layers (HML) were introduced by Halpern et al. in 2011 [2] for general hyperbolic operators. This method is based on an extrapolation of solutions using first-order layers. For constant coefficient problems, its numerical performances are comparable to those of Bérenger's PML [1], while preserving the strong well-posedness of the hyperbolic system. In homogeneous media, the PML are nonreflecting and absorbing. This is no longer true in heterogeneous media. In that case, other methods may become more attractive. Numerical experiments involving the propagation of 2D acoustic waves in an inhomogeneous medium show smaller amplitude reflections at the interface for the HML.

**Introduction**

The numerical simulation of wave propagation in unbounded media occurs in numerous industrial applications, such as radar detection or seismic imaging. In these contexts, the use of Perfectly Matched Layer (PML), introduced by Bérenger for the Maxwell system, has rapidly become the state-of-the-art technique to perform such simulations [1]. The domain of interest is surrounded with a damping layer, where the incident waves should be absorbed without reflections for any incidence angle. The original unknowns of the hyperbolic system related to the wave equation are split into non-physical unknowns, and a damping factor is introduced in the resulting equations. Years of successful applications due to its efficiency and its ease of implementation have followed Bérenger's discovery.

In the context of variable background coefficients, the model can still be used, but needs some care in the implementation and the mathematical analysis, as the reflectivity of the layer becomes non negligible, and strong well-posedness can be lost [3]. In this study, we are interested in a new layer method, introduced by Halpern et al [2], named Harmoniously Matched Layers (HML). This method is designed to keep the well-posedness of the original hyperbolic system by using a classical first-order damping layer (named SMART layer in the sequel). The reflectivity of the layer is controlled by an extrapolation technique which annihilates first-order reflections in the high frequency regime. The aim of this study is to compare PML and HML methods in the simple case of 2D acoustic waves propagation.

**PML and HML formulation for 2D acoustic wave propagation within the subsurface**

Consider the first-order velocity-stress formulation for the 2D propagation of acoustic waves in $\Omega = [0, L]^2 \subset \mathcal{R}^2$, with variable density $\rho(x_1, x_2)$ and velocity $c(x_1, x_2)$[1],

$$\left\{\begin{array}{rclrcl} \partial_t u - \dfrac{1}{\rho}\nabla p & = & 0 & u(x_1,x_2,0) & = & 0, \\ \partial_t p - \rho c^2 \mathrm{div} u & = & 0, & p(x_1,x_2,0) & = & p_0(x_1,x_2). \end{array}\right. \tag{1}$$

Here, $u = (u_1, u_2)$ is the displacement velocity vector, $p$ is the pressure wavefield, and $p_0$ is the initial pressure wavefield. The PML equations associated with system (1) are defined on $\Omega_l = [-l; L+l]^2 \subset \mathcal{R}^2$

$$\left\{\begin{array}{l} \partial_t u_j - \dfrac{1}{\rho}\partial_j(p_1 + p_2) + \sigma_j u_j = 0 \\ \partial_t p_j - \rho c^2 \partial_j u_j + \sigma_j p_j = 0 \end{array}\right. \tag{2}$$

where $p_j$ denote the non-physical split pressure wavefields, and $\sigma_j(x_j)$ are the absorbing coefficients non zero only in $\Omega_l \backslash \Omega$. This system is weakly well posed: for piecewise continuous velocity and impedance, energy estimates with one loss of derivatives have been obtained in [4]. The SMART equations add to the operator the zero order perturbation defined by the projectors on the relevant eigenspaces in each direc-

[1]For convenience, we restrict our notations to the square domain case. Extension to rectangular domains is straightforward.

tion.

$$
\begin{cases}
\partial_t u_j - \frac{1}{\rho}\partial_j p + \frac{1}{1+\rho^2 c^2} F_j(u,p) \\
\partial_t p - \rho c^2 \mathrm{div} u + \frac{\rho c}{1+\rho^2 c^2} \displaystyle\sum_{j=1,2} F_j(u,p) = 0. \\
F_j(u,p) = \left[\sigma_j^+ \left(u_j + \rho c p\right) + \sigma_j^- \left(u_j - \rho c p\right)\right]
\end{cases}
\tag{3}
$$

where $\sigma_j^+ = (\sigma_j)_{|[L;L+l]}, \sigma_j^- = (\sigma_j)_{|[-l;0]}$ (see [2] for a detailed derivation of these equations). The SMART equations keep the strong well-posedness of the original system (1). Denote $U(\sigma) = (u_1, u_2, p)$ the solution of the SMART system for an absorption $\sigma = (\sigma_1, \sigma_2)$. The HML strategy consists in computing an extrapolation $U_{HML}$ such that

$$U_{HML} = U(2\sigma) - 2U(\sigma). \tag{4}$$

**Numerical study of reflectivity in a varying medium**

Consider $\Omega$ and $\Omega_l$ such that $L = l = 5$. The density $\rho$ is taken constant equal to 1. The non constant velocity $c$ is

$$c(x_1, x_2) = 2 + \sin(3(x_1 - L)). \tag{5}$$

The absorption coefficients $\sigma_j$ are the usual order 3 polynomials (exact formula are given for instance in [4]). Homogeneous Dirichlet boundary conditions are imposed at the boundary of $\Omega_l$. We choose the initial condition $p_0$ such that, for $\mathbf{x} = (x_1, x_2)$, $\mathbf{x}_C = (4.5, 2)$, $\mathbf{v} = (1, -1)$, $k = 3$, $r = 0.8$:

$$
\begin{array}{l}
p_0(x_1, x_2) = f(x_1, x_2)\chi_{\|\mathbf{x}-\mathbf{x}_C\| \le r} \\
f(x_1, x_2) = \cos^2\left(\pi \frac{\|\mathbf{x}-\mathbf{x}_C\|}{r}\right) \cos\left(k\pi \frac{\mathbf{v}.(\mathbf{x}-\mathbf{x}_C)}{r}\right)
\end{array}
\tag{6}
$$

A "beam" centered on $x_C$ propagates along the line $x = -z$ and hits the absorbing layer with an non-normal incidence angle. We present on figure 1 the resulting pressure wavefield at time $t = 0.75$ s and the difference of the PML, SMART and HML solution with the exact solution. The PML layer generates a reflected wave at the interface with the interest domain, whose infinity norm is close to $10^{-4}$. The reflection generated by the SMART layer reaches $10^{-3}$ in infinity norm. Conversely, the extrapolation technique used for the HML method yield a significant decrease of the reflection, which only reaches $10^{-7}$ in infinity norm in this case.

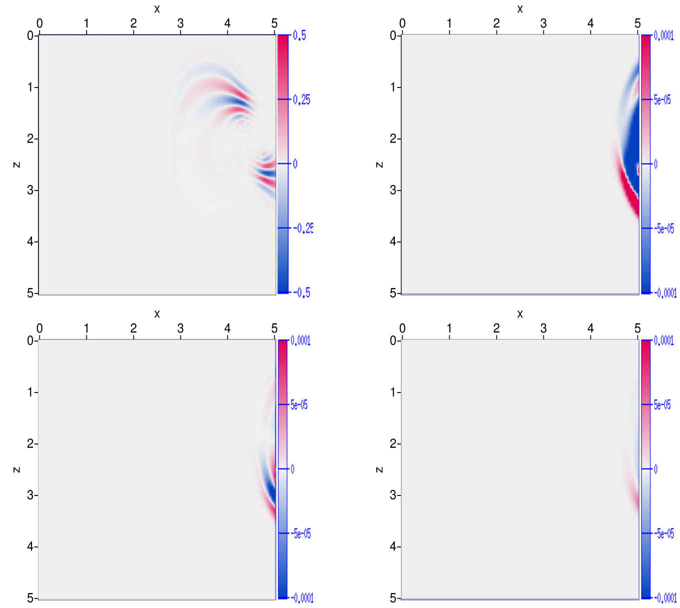

Figure 1. Exact pressure wavefield (top left). Difference between exact and results from SMART (top right), PML (bottom left), HML (bottom right).

**Conclusion and perspectives**

This preliminary experiment demonstrates that, for variable coefficients problems, the HML method can yield improvements in terms of reflectivity compared to the PML method. Another advantage is that the HML formulation keeps the well-posedness of the initial set of equations, which may yield more robust absorbing layer formulations for the simulation of wave propagating in elastic and anisotropic medias. The mathematical analysis of these models is undergoing.

**References**


[1] J.-P. Berenger, *A perfectly matched layer for absorption of electromagnetic waves*, Journal of Computational Physics, 114 (1994), pp. 185–200.

[2] L. Halpern, S. Petit-Bergez, and J. Rauch, *The Analysis of Matched Layers*, Confluentes Mathematici, 3 (2011), pp. 159–236.

[3] J. Hesthaven, *On the analysis and construction of perfectly matched layers for the linearized euler equations*, Journal of Computational Physics, 142 (1998), pp. 129–147.

[4] L. Métivier, *Utilisation des équations Euler-PML en milieu hétérogène borné pour la résolution d'un problème inverse en géophysique.*, ESAIM: Proc., 27 (2009), pp. 156–170.